General Relativity as a constrained Gauge Theory

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Abstract

The formulation of General Relativity presented in [1] and the Hamiltonian formulation of Gauge theories described in [2] are made to interact. The resulting scheme allows to see General Relativity as a constrained Gauge theory.

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1 Geometrical preliminaries

The recent developments in the study of $\mathcal{J}$-bundles [3, 4, 5] allowed to build a first-order frame formulation of General Relativity [1] on the one hand and a regular Hamiltonian formulation of gauge theories [2] on the other. The interaction of the two aspects will allow to deduce General Relativity as a constrained variational problem for a Hamiltonian function of a $SO(1,3)$-gauge theory.
For this purpose, the geometrical frameworks contained in [1] and [2] will be briefly revised: only the main results will be exposed, referring the reader to the cited works for the details and the proofs.
First of all, the purely frame formulation of General relativity described in [1] will be considered.
Let M be a 4-dimensional space–time manifold, allowing a metric g with signature $(1,3)$. Let $P \to M$ be a principal fiber bundle having structural group $G = SO(1,3)$ and $L(M) \to M$ be the co–frame bundle over M.
According to the gauge natural bundle framework (see [6] and references therein), the configuration space of the theory is the $GL(4,\Re)$-bundle $\pi : \mathcal{E} \to M$, associated

with $P \times_M L(M)$ through the left-action

$$\lambda : (SO(1,3) \times GL(4,\Re)) \times GL(4,\Re) \to GL(4,\Re), \qquad \lambda(\Lambda, J; X) = \Lambda \cdot X \cdot J^{-1} \tag{1.1}$$

The space $\mathcal{E}$ can be referred to local coordinates x^i, e^μ_i ($i, \mu = 1, \ldots, 4$), subject to the following transformation laws

$$\bar{x}^i = \bar{x}^i(x^j), \qquad \bar{e}^\mu_j = e^\sigma_i \Lambda^\mu{}_\sigma(x) \frac{\partial x^i}{\partial \bar{x}^j} \tag{1.2}$$

with $\Lambda^\mu{}_\sigma(x) \in SO(1,3)\ \forall\, x \in M$.
The dynamical fields of the theory are sections $\gamma : x^i \to (x^i, e^\mu_j(x^i))$ of the bundle $\mathcal{E} \to M$. Every section γ can be thought as a family of local sections of $L(M) \to M$, glued to each another by Lorentz transformations. Any such a section induce a corresponding metric on M, defined as $g_{ij}(x^k) := \eta_{\mu\nu} e^\mu_i(x^k) e^\nu_j(x^k)$, where $\eta_{\mu\nu} = \eta^{\mu\nu} := diag(-1, 1, 1, 1)$.
Moreover, let $\mathcal{C} \to M$ denote the space of principal connections on P, consisting in the quotient bundle $j_1(P, M)/G$. A set of local coordinates over $\mathcal{C}$ is provided by the functions $x^i, \omega_i{}^{\mu\nu}$ ($\mu < \nu$), subject to the following transformation laws

$$\bar{x}^i = \bar{x}^i(x^j), \qquad \bar{\omega}_i{}^{\mu\nu} = \Lambda^\mu{}_\sigma(x) \Lambda^\nu{}_\gamma(x) \frac{\partial x^j}{\partial \bar{x}^i} \omega_j{}^{\sigma\gamma} - \Lambda^\eta{}_\sigma(x) \frac{\partial \Lambda^\mu{}_\eta(x)}{\partial x^h} \frac{\partial x^h}{\partial \bar{x}^i} \eta^{\sigma\nu} \tag{1.3}$$

where $\Lambda^\mu{}_\nu(x) \in SO(1,3)\ \forall\, x \in M$ and $\Lambda_\sigma{}^\nu := \left(\Lambda^{-1}\right)^\nu{}_\sigma = \Lambda^\alpha{}_\beta \eta_{\alpha\sigma} \eta^{\beta\nu}$.
The velocity space of the theory is provided by the first $\mathcal{J}$-bundle of $\pi : \mathcal{E} \to M$. It is built similarly to an ordinary jet–bundle, but the first order contact between sections is calculated with respect to the exterior covariant differential (compare with [1] for the details). As far as this paper is concerned it is only needed to know that:
1. The bundle $\mathcal{J}(\mathcal{E})$ has all the properties of standard jet-bundles (compare with [4]), such as contact 1-forms, raising of sections and vector fields.
2. The bundle $\mathcal{J}(\mathcal{E})$ is diffeomorphic to the fiber product $\mathcal{E} \times_M \mathcal{C}$ over M. $\mathcal{J}(\mathcal{E})$ can be then referred to local coordinates $x^i, e^\mu_i, \omega_j{}^{\mu\nu}$ as above. In such coordinates, a section $\gamma : M \to \mathcal{J}(\mathcal{E})$, $\gamma : x^k \to (x^k, e^\mu_i(x^k), \omega_i{}^{\mu\nu}(x^k))$, is said holonomic — or kinematically admissible — if the quantities $\omega_i{}^{\mu\nu}(x^k)$ are the coefficients of the spin connection generated by the metric $g_{ij}(x^k) = \eta_{\mu\nu} e^\mu_i(x^k) e^\nu_j(x^k)$.
3. The variational problem built on $\mathcal{J}(\mathcal{E})$ through the 4-form

$$\Theta = \frac{1}{4} \epsilon^{qpij} \epsilon_{\mu\nu\lambda\sigma} e^\mu_q e^\nu_p \left(d\omega_i{}^{\lambda\sigma} \wedge ds_j + \omega_j{}^\lambda{}_\eta \omega_i{}^{\eta\sigma} ds \right) \tag{1.4}$$

with $ds := dx^1 \wedge \ldots \wedge dx^4$ and $ds_j := \frac{\partial}{\partial x^j} \rfloor ds$, provides the following field equations for critical sections $\gamma : x^k \to (x^k, e^\mu_i(x^k), \omega_i{}^{\mu\nu}(x^k))$

$$\epsilon^{qpij} \epsilon_{\mu\nu\lambda\sigma} e^\mu_q \left(\frac{\partial e^\nu_p}{\partial x^j} + \omega_j{}^\nu{}_\rho e^\rho_p \right) = 0 \tag{1.5a}$$

$$\frac{1}{2}\epsilon^{qpij}\epsilon_{\mu\nu\lambda\sigma}e^\mu_q\left(\frac{\partial{\omega_i}^{\lambda\sigma}}{\partial x^j}+{\omega_j}^\lambda{}_\eta{\omega_i}^{\eta\sigma}\right)=0 \tag{1.5b}$$

deriving from the Euler–Lagrange equations

$$\gamma^*(X \rfloor d\Theta)=0 \qquad \forall X\in D^1(\mathcal{J}(\mathcal{E}))$$

associated with the form Θ through usual vanishing boundary conditions.
Eqs. (1.5a) ensure the kinematic admissibility of the critical sections, expressed as

$$\frac{\partial e^\nu_p}{\partial x^j}(x)-\frac{\partial e^\nu_j}{\partial x^p}(x)={\omega_p}^\nu{}_\rho(x)e^\rho_j(x)-{\omega_j}^\nu{}_\rho(x)e^\rho_p(x)$$

allowing us to identify the components ${\omega_p}^\nu{}_\rho(x)$ with the coefficients of the spin–connection associated with the metric $g_{ij}(x)=\eta_{\mu\nu}e^\mu_i(x)e^\nu_j(x)$.
Taking the previous result into account, eqs. (1.5b) are equivalent to Einstein equations (provided that $\det(e^\mu_i)\neq 0$), written in the form

$$\frac{1}{4}\epsilon^{qpij}\epsilon_{\mu\nu\lambda\sigma}e^\mu_p(x){R_{ji}}^{\lambda\sigma}(x)=0$$

where

$${R_{ji}}^{\lambda\sigma}(x)=\frac{\partial{\omega_i}^{\lambda\sigma}}{\partial x^j}(x)-\frac{\partial{\omega_j}^{\lambda\sigma}}{\partial x^i}(x)+{\omega_j}^\lambda{}_\eta(x){\omega_i}^{\eta\sigma}(x)-{\omega_i}^\lambda{}_\eta(x){\omega_j}^{\eta\sigma}(x)$$

denotes the curvature tensor of the metric g.
As a result, the geometrical framework $(\mathcal{J}(\mathcal{E}),\Theta)$ is the natural setting to build a variational first–order purely frame (or, equivalently, a zero–order frame–affine) formulation of General Relativity.
On the other hand, the Hamiltonian framework for gauge theories has been described in [2]. Given a principal bundle $P\to M$, which takes the gauge-invariance into account, it is possible to create a geometrical framework where gauge theories can be described as *non-singular* Lagrangian and Hamiltonian theories. The argument consists in taking the space $\mathcal{C}\to M$ of connection 1-forms over P into account and building its first $\mathcal{J}$-bundle. This structure is best described in [4]; in particular it is there shown how the components ${R_{ij}}^{\mu\nu}$ of the curvature forms can be chosen as coordinates over the fibers of $\mathcal{J}(\mathcal{C})$. The use of the $\mathcal{J}$-bundle allows to build a *regular* Lagrangian theory.
From now on, a review of the argument is proposed. Nevertheless, it will be adapted to the current situation, where the gauge group is $SO(1,3)$.
1. A set of local coordinates on $\mathcal{J}(\mathcal{C})$ is provided by the functions $x^i,{\omega_j}^{\mu\nu},{R_{ij}}^{\mu\nu}$ ($\mu<\nu$, $i<j$), subject to the transformation laws (1.3) and

$${\bar{R}_{ij}}{}^{\mu\nu}={R_{rs}}^{\lambda\sigma}\frac{\partial x^r}{\partial\bar{x}^i}\frac{\partial x^s}{\partial\bar{x}^j}\Lambda^\mu{}_\lambda\Lambda^\nu{}_\sigma$$

2. The Hamiltonian theory is built considering the module $\Lambda^4(\mathcal{C})$ of 4-forms over $\mathcal{C}$ and its sub–bundles $\Lambda^4_r(\mathcal{C})$ consisting of those forms vanishing when r of their

arguments are vertical vectors. The whole argument is described in a quite general setup in [2].
In particular, the following submodules are taken into account:

$$\Lambda^4_1(\mathcal{C}) := \{\alpha \in \Lambda^4(\mathcal{C}) : \alpha = p(\alpha)\, ds\} \tag{1.6}$$

and

$$\Lambda^4_2(\mathcal{C}) := \{\alpha \in \Lambda^4(\mathcal{C}) : \alpha = p(\alpha)\, ds + \frac{1}{2}\Pi^{ji}{}_{\mu\nu}(\alpha)\, d\omega_i{}^{\mu\nu} \wedge ds_j\} \tag{1.7}$$

The sets of functions $\{x^i, \omega_i{}^{\mu\nu}, p\}$ and $\{x^i, \omega_i{}^{\mu\nu}, p, \Pi^{ij}{}_{\mu\nu}\}$ $(\mu < \nu)$ form systems of local coordinates on $\Lambda^4_1(\mathcal{C})$ and $\Lambda^4_2(\mathcal{C})$ respectively.
The bundle $\Lambda^4_2(\mathcal{C})$ is endowed with the canonical Liouville 4-form, locally expressed as

$$\Theta := p\, ds + \frac{1}{2}\Pi^{ji}{}_{\mu\nu}\, d\omega_i{}^{\mu\nu} \wedge ds_j \tag{1.8}$$

Being the bundle $\Lambda^4_1(\mathcal{C})$ a vector sub-bundle of $\Lambda^4_2(\mathcal{C})$, the quotient bundle $\Lambda^4_2(\mathcal{C})/\Lambda^4_1(\mathcal{C})$ can be considered. The latter has the nature of a vector bundle over $\mathcal{C}$ and is locally described by the system of coordinates $x^i, \omega_i{}^{\mu\nu}, \Pi^{ij}{}_{\mu\nu}$, while the canonical projection makes $\pi : \Lambda^4_2(\mathcal{C}) \to \Lambda^4_2(\mathcal{C})/\Lambda^4_1(\mathcal{C})$ into an affine bundle.
3. The *phase space* of the theory is defined as the vector sub–bundle $\Pi(\mathcal{C}) \subset \Lambda^4_2(\mathcal{C})/\Lambda^4_1(\mathcal{C})$ consisting of those elements satisfying

$$\Pi^{ij}{}_{\mu\nu}(z) = -\Pi^{ji}{}_{\mu\nu}(z) \tag{1.9}$$

A local system of coordinates for $\Pi(\mathcal{C})$ is provided by $x^i, \omega_i{}^{\mu\nu}, \Pi^{ij}{}_{\mu\nu}$ $(i < j, \mu < \nu)$, subject to the transformation laws (1.3) together with (see [2])

$$\bar{\Pi}^{pq}{}_{\lambda\sigma} = det\left(\frac{\partial x^h}{\partial \bar{x}^k}\right)\Pi^{ij}{}_{\mu\nu}\Lambda_\lambda{}^\mu\Lambda_\sigma{}^\nu\frac{\partial \bar{x}^q}{\partial x^j}\frac{\partial \bar{x}^p}{\partial x^i} \tag{1.10}$$

Besides, being $\Pi(\mathcal{C})$ a vector sub–bundle, the immersion $i : \Pi(\mathcal{C}) \to \Lambda^4_2(\mathcal{C})/\Lambda^4_1(\mathcal{C})$ is well defined and is locally represented by eq. (1.9) itself.
4. The pull–back bundle $\hat{\pi} : \mathcal{H}(\mathcal{C}) \to \Pi(\mathcal{C})$ defined by the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}(\mathcal{C}) & \xrightarrow{\hat{i}} & \Lambda^4_2(\mathcal{C}) \\ {\scriptstyle \hat{\pi}}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\ \Pi(\mathcal{C}) & \xrightarrow{i} & \Lambda^4_2(\mathcal{C})/\Lambda^4_1(\mathcal{C}) \end{array} \tag{1.11}$$

will now be taken into account. The latter has the nature of an affine bundle over the phase space. Every section $h : \Pi(\mathcal{C}) \to \mathcal{H}(\mathcal{C})$ is called a *Hamiltonian section*, and is locally described in the form:

$$h : p = -H(x^i, \omega_i{}^{\mu\nu}, \Pi^{ij}{}_{\mu\nu}) - \frac{1}{16}\Pi^{ij}{}_{\mu\nu}\omega_i{}^{\lambda\sigma}\omega_j{}^{\rho\beta}C^{\mu\nu}_{\rho\beta\,\lambda\sigma} \tag{1.12}$$

where $C^{\mu\nu}_{\rho\beta\,\lambda\sigma}$ are the structure coefficients of the group $SO(1,3)$. In accordance with the literature, the function $H(x^i, \omega_i{}^{\mu\nu}, \Pi^{ij}{}_{\mu\nu})$ is called the *Hamiltonian* of the system.
The presence of the immersion $\hat{i} : \mathcal{H}(\mathcal{C}) \to \Lambda^4_2(\mathcal{C})$, endows $\mathcal{H}(\mathcal{C})$ with the canonical 4-form $\hat{i}^*(\Theta)$, locally expressed as in eq. (1.8). The latter will be simply denoted as Θ and will be called the Liouville form on $\mathcal{H}(\mathcal{C})$.
5. The assignment of the Hamiltonian section allows to perform the pull-back of the Liouville form on $\mathcal{H}(\mathcal{C})$ to the phase space $\Pi(\mathcal{C})$. The result is a Hamiltonian dependent 4-form

$$\Theta_h := h^*(\Theta) = -H(x^i, \omega_i{}^{\mu\nu}, \Pi^{ij}{}_{\mu\nu})\, ds - \frac{1}{2}\Pi^{ij}{}_{\mu\nu}\left(d\omega_i{}^{\mu\nu} \wedge ds_j + \frac{1}{8}\omega_i{}^{\lambda\sigma}\omega_j{}^{\rho\beta}C^{\mu\nu}_{\rho\beta\,\lambda\sigma}\, ds\right) \tag{1.13}$$

Starting from the coordinate transformations (1.3) and (1.10), it is easily seen that the form (1.13) is covariant. The reader is referred to [2] for a more general explanation.
The free variational problem performed on $\Pi(\mathcal{C})$ through the form (1.13) provides the Hamilton–De Donder equations

$$\gamma^*(X \rfloor d\Theta_h) = 0 \qquad \forall X \in D^1(\Pi(\mathcal{C}))$$

yielding final field equations for critical sections $\gamma : x^k \to (x^k, \omega_i{}^{\mu\nu}(x^k), \Pi^{ij}{}_{\mu\nu}(x^k))$ of the form

$$-\frac{\partial H}{\partial \Pi^{ij}{}_{\alpha\beta}} - \frac{\partial \omega_i{}^{\alpha\beta}}{\partial x^j} + \frac{\partial \omega_j{}^{\alpha\beta}}{\partial x^i} - \frac{1}{4}\omega_i{}^{\nu\mu}\omega_j{}^{\rho\lambda}C^{\alpha\beta}_{\rho\lambda\,\nu\mu} = 0 \tag{1.14a}$$

$$-\frac{\partial H}{\partial \omega_i{}^{\mu\nu}} - \frac{\partial \Pi^{ji}{}_{\mu\nu}}{\partial x^j} + \frac{1}{4}\Pi^{ji}{}_{\lambda\sigma}\omega_j^{\gamma\alpha}C^{\lambda\sigma}_{\gamma\alpha\,\mu\nu} = 0 \tag{1.14b}$$

6. The acquired regularity of the Hamiltonian allows to build a non–singular inverse Legendre transformation $Leg^{-1} : \Pi(\mathcal{C}) \to \mathcal{J}(\mathcal{C})$, described as

$$R_{st}{}^{\alpha\beta} = \frac{\partial H}{\partial \Pi^{st}{}_{\alpha\beta}} \tag{1.15}$$

The Lagrangian associated with the Hamiltonian H can be obtained as

$$L(x^i, \omega_i{}^{\mu\nu}, R_{ij}{}^{\lambda\sigma}) = \frac{1}{4}\Pi^{ij}{}_{\lambda\sigma}R_{ij}{}^{\lambda\sigma} - H(x^i, \omega_i{}^{\mu\nu}, \Pi^{ij}{}_{\lambda\sigma}(x^i, \omega_i{}^{\mu\nu}, R_{ij}{}^{\mu\nu})) \tag{1.16}$$

The latter gives rise to the Legendre transformation $leg : \mathcal{J}(\mathcal{C}) \to \Pi(\mathcal{C})$, expressed as

$$\Pi^{st}{}_{\alpha\beta} = \frac{\partial L}{\partial R_{st}{}^{\alpha\beta}} \tag{1.17}$$

Taking eqs. (1.15), (1.16) and (1.17) into account, the Lagrangian counterpart of the field equations (1.14) can be expressed as

$$R_{ij}{}^{\alpha\beta} = -\frac{\partial \omega_i{}^{\alpha\beta}}{\partial x^j} + \frac{\partial \omega_j{}^{\alpha\beta}}{\partial x^i} - \frac{1}{4}\omega_i{}^{\nu\mu}\omega_j{}^{\rho\lambda}C^{\alpha\beta}_{\rho\lambda\,\nu\mu} \tag{1.18a}$$

$$\frac{\partial L}{\partial \omega_i{}^{\mu\nu}} - D_j\frac{\partial L}{\partial R_{ji}{}^{\mu\nu}} = 0 \tag{1.18b}$$

D_j denoting covariant differentiation. In particular, eqs. (1.18a) ensure the kinematic admissibility of the critical sections, so that eqs. (1.18b) represent actual Lagrange equations in gauge theory.

2 Gauge Theory and General Relativity

The actual object of the paper is to study the theory arising from the assignment of the following Hamiltonian function

$$H = \Pi^{ij}{}_{\mu\nu}\Pi^{pq}{}_{\lambda\sigma}\eta^{\mu\lambda}\eta^{\nu\sigma}\epsilon_{ijpq} \tag{2.1}$$

on the phase space $\Pi(\mathcal{C})$. ϵ_{ijpq} denotes the Levi–Civita permutation symbol.
The Hamiltonian (2.1) induces the corresponding Hamiltonian 4-form (1.13). Taking the explicit expression of the structure coefficients $C^{\mu\nu}_{\rho\beta\,\lambda\sigma}$ for the group $SO(1,3)$ into account, we have

$$\Theta_h = -H ds - \frac{1}{2}\Pi^{ij}{}_{\mu\nu}\left(d\omega_i{}^{\mu\nu}\wedge ds_j + \omega_j{}^{\mu}{}_{\lambda}\omega_i{}^{\lambda\nu}ds\right) \tag{2.2}$$

The properties of the Hamiltonian (2.1) are described by the following

Proposition 2.1 *The Hamiltonian* (2.1) *is regular.*

PROOF. One could prove the above statement through a direct calculation. More simply, it is easily seen that the inverse Legendre transformation, induced by Hamiltonian (2.1),

$$R_{st}{}^{\alpha\beta} = \frac{\partial H}{\partial \Pi^{st}{}_{\alpha\beta}} = 8\Pi^{pq}{}_{\lambda\sigma}\eta^{\alpha\lambda}\eta^{\beta\sigma}\epsilon_{stpq} \tag{2.3}$$

is bijective, being its inverse function (the Legendre transformation) provided by

$$\Pi^{ij}{}_{\lambda\sigma} = \frac{1}{32}R_{st}{}^{\alpha\beta}\eta_{\alpha\lambda}\eta_{\beta\sigma}\epsilon^{stij} \tag{2.4}$$

□

A comparison with eqs. (1.16), (2.3) and (2.4) allows to obtain the expression for the Lagrangian L associated with the Hamiltonian (2.1):

$$L = \frac{1}{256}R_{st}{}^{\alpha\beta}R_{ij}{}^{\lambda\sigma}\eta_{\alpha\lambda}\eta_{\beta\sigma}\epsilon^{stij}$$

Now, let us define a map $i : \mathcal{J}(\mathcal{E}) \to \Pi(\mathcal{C})$, locally expressed as

$$\begin{cases} x^i = x^i \\ {\omega_i}^{\mu\nu} = {\omega_i}^{\mu\nu} \\ {\Pi^{ij}}_{\lambda\sigma} = -\frac{1}{2} e^\mu_q e^\nu_p \epsilon^{qpij} \epsilon_{\mu\nu\lambda\sigma} \end{cases} \tag{2.5}$$

Note that, according to the transformation laws (1.2) and (1.10), the map (2.5) is well defined.
We have the following result

Proposition 2.2 *The map* $i : \mathcal{J}(\mathcal{E}) \to \mathcal{H}(\mathcal{C})$*, defined by eq.* (2.5)*, is an immersion.*

PROOF. A direct calculation shows that

$$\frac{\partial {\Pi^{ij}}_{\lambda\sigma}}{\partial e^\alpha_k} = -e^\mu_p \epsilon^{kpij} \epsilon_{\alpha\mu\lambda\sigma}$$

The kernel of the differential of the map (2.5) is determined by the condition

$$\frac{\partial {\Pi^{ij}}_{\lambda\sigma}}{\partial e^\alpha_k} V^\alpha_k = 0 \quad \Leftrightarrow \quad e^\mu_p \epsilon^{kpij} \epsilon_{\alpha\mu\lambda\sigma} V^\alpha_k = 0$$

to be solved in the unknowns V^α_k.
A saturation with e^λ_i and e^ρ_j yields

$$e^\lambda_i e^\rho_j e^\mu_p \epsilon^{kpij} \epsilon_{\alpha\mu\lambda\sigma} V^\alpha_h e^h_\beta e^\beta_k = 0 \quad \Leftrightarrow \quad e\, \epsilon^{\beta\mu\lambda\rho} \epsilon_{\alpha\mu\lambda\sigma} V^\alpha_h e^h_\beta = 0$$

which is equivalent to:

$$\delta^\beta_\alpha \delta^\rho_\sigma V^\alpha_h e^h_\beta - \delta^\rho_\alpha \delta^\beta_\sigma V^\alpha_h e^h_\beta = 0$$

The above expression can be split into two cases:
i) $\rho \neq \sigma \quad \Rightarrow \quad V^\rho_h e^h_\sigma = 0$
ii) $\rho = \sigma \quad \Rightarrow \quad V^\alpha_h e^h_\alpha - V^\sigma_h e^h_\sigma = 0$ (the index σ is not summed). Summing these four equations ($\sigma = 1, \ldots, 4$) one gets $3V^\alpha_h e^h_\alpha = 0$ (summed over α) and therefore also $V^\sigma_h e^h_\sigma = 0 \quad \forall\, \sigma = 1, \ldots, 4$ (σ not summed).
We thus get that $V^\rho_h e^h_\sigma = 0 \quad \forall\, \sigma, \rho = 1, \ldots, 4$. A final saturation with e^σ_i yields $V^\nu_h = 0\ \forall\, h, \nu = 1, \ldots, 4$ as a result. The application i is therefore an immersion.
$\square$

As a consequence of Proposition 2.2, the space $\mathcal{J}(\mathcal{E})$ has the nature of an immersed submanifold [7] of $\Pi(\mathcal{C})$, fibered over $\mathcal{C}$. A simple calculation shows that the pull-back $\Theta = i^*(\Theta_h)$ of the form (2.2) by means of the map $i : \mathcal{J}(\mathcal{E}) \to \Pi(\mathcal{C})$ is the very same 4-form (1.4) leading the regular variational approach to General Relativity on $\mathcal{J}(\mathcal{E})$.
The argument is based on the following

Proposition 2.3 *The pull-back of the Hamiltonian* (2.1) *through the map i vanishes identically, i.e. $i^*(H) = 0$.*

PROOF.

$$
\begin{aligned}
i^*(H) =& \frac{1}{4} e^\xi_s e^\eta_t \epsilon_{\xi\eta\mu\nu} \epsilon^{stij} e^\alpha_h e^\beta_k \epsilon_{\alpha\beta\lambda\sigma} \epsilon^{hkpq} \eta^{\mu\lambda} \eta^{\nu\sigma} \epsilon_{ijpq} = \\
=& \frac{1}{2} e^\xi_s e^\eta_t e^\alpha_h e^\beta_k \epsilon^{stij} \epsilon_{\xi\eta\mu\nu} \epsilon_{\alpha\beta\lambda\sigma} \left(\delta^h_i \delta^k_j - \delta^k_i \delta^h_j \right) \eta^{\mu\lambda} \eta^{\nu\sigma} = \\
=& e^\xi_s e^\eta_t e^\alpha_h e^\beta_k \epsilon^{sthk} \epsilon_{\xi\eta\mu\nu} \epsilon_{\alpha\beta\lambda\sigma} \eta^{\mu\lambda} \eta^{\nu\sigma} = e\ \epsilon^{\xi\eta\alpha\beta} \epsilon_{\alpha\beta\lambda\sigma} \epsilon_{\xi\eta\mu\nu} \eta^{\mu\lambda} \eta^{\nu\sigma} = \\
=& 2e\ \left(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu \right) \epsilon_{\alpha\beta\lambda\sigma} \eta^{\mu\lambda} \eta^{\nu\sigma} = 4e\ \epsilon_{\mu\nu\lambda\sigma} \eta^{\mu\lambda} \eta^{\nu\sigma} = 0
\end{aligned}
$$

□

Summing all up, we have proved that the formulation of General Relativity proposed in [1] is deducible from a constrained variational problem in a $SO(1,3)$-gauge theory, described by the Hamiltonian (2.1) and the constraint (2.5).
In this respect, we notice that, although the map i is not one–to–one with its image, the constrained variational problem is well defined; this is due to the fact that if $z_1, z_2 \in \mathcal{J}(\mathcal{E})$ and $i(z_1) = i(z_2)$, then we have $i^*_{z_1}(\Theta_h) = i^*_{z_2}(\Theta_h)$.

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